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On Weakly Almost Generalized 2-Absorbing Sub-modules of Modules

Arwa E. Ashour[1,*]

[1]Department of Mathematics, Faculty of Science, Islamic University of Gaza, Gaza Strip, Palestine

*** Corresponding author**
e-mail address: arashour@iugaza.edu.ps

Abstract

Let M be a module over a commutative ring R with non-zero identity. A proper sub-module N of M is called weakly almost generalized 2-absorbing (denoted by *WAG2*-absorbing) sub-module, if for $a,b \in R$ and $m \in M$ with $0 \neq abm \in N$, either $ab \in \sqrt{(N:M)}$ or $a^i m \in N$ or $b^j m \in N$ for some positive integers i and j. We study the relation between *WAG2*-absorbing sub-modules and primary, weak primary and weakly primary sub-modules. Also, we study the behavior of $\sqrt{(N:M)}$, when N is *WAG2*-absorbing sub-module. Moreover, the *WAG2*-absorbing sub-modules when $R = R_1 \oplus R_2$ are characterized.

Keywords:
Primary submodules,
Weak primary submodules,
Weakly primary submodules,
Almost generalized 2-absorbing submodules,
Weakly almost generalized 2-absorbing submodules,
Colon ideal of a submodule,
Radical ideal of a submodule.

1. Introduction:

Throughout this paper, all rings are assumed to be commutative with non-zero identity and all modules are unitary. Furthermore, we consider R to be a commutative ring with identity and M an R-module,
The colon ideal of a submodule N of M is considered to be $(N:M) = \{r \in R \mid rm \subseteq N\}$. Moreover, $\sqrt{(N:M)} = \{r \in R \mid r^n m \subseteq N, \text{ for some positive integer } n\}$. will be called the radical ideal of N.
New concepts related to primary and weakly primary submodules were introduced and studied by (Chinwarakorn and Pianskool, 2015). These are the concepts of *AG2*-absorbing submodules and *WAG2*-absorbing submodules. The important of these concepts is the result that was proved in (Chinwarakorn and Pianskool, 2015), which says that the intersection of each distinct primary (weakly primary) submodules of an R-module M is *AG2*-absorbing (*WAG2*-absorbing) submodule.

According to (Darani and Soheilnia, 2011), a proper submodule N of M is called weakly 2-absorbing, if for $a,b \in R$ and $m \in M$ with $0 \neq abm \in N$, either $ab \in (N:M)$ or $am \in N$ or $bm \in N$. It is clear that, as Chinwarakern and Pianskool clarified in (Chinwarakorn and Pianskool, 2015), the concept of *WAG2*-absorbing submodules is a generalization of the concept of weakly 2-absorbing submodules.
In 2016, Moradi and Azizi proved some important results concerning weakly 2-absorbing submodules and their colon ideals and they studied some properties of weakly 2-absorbing submodules when $R = R_1 \oplus R_2$. These results concerning weakly 2-absorbing submodules that were proved in (Moradi and Azizi, 2016) inspired us to make a generalization for them on *WAG2*-absorbing submodules

In Section 2 we recall some definitions concerning primary submodules such as weak primary and weakly primary submodules and study the relation between these submodules and *WAG2*-absorbing submodules. We prove that the concepts of weak primary and *AG2*-absorbing submodules are equivalent for a submodule if its radical ideal is prime.

Section 3, study the behavior of the radical of a submodule and find the answer of the following question:

Question: Suppose that L is a *WAG2*-absorbing ideal of a ring R and $0 \neq IJK \subseteq L$ for some ideals *I, J* and cyclic ideal K of R. Does it imply that $IJ \subseteq \sqrt{L}$ or $IK \subseteq \sqrt{L}$ or $JK \subseteq \sqrt{L}$. Solving this question was an attempt to solve a question that make a generalization of a question asked by Badawi and Darani in (Badawi and Darani, 2013) and answered by Moradi and Azizi in (Moradi and Azizi, 2016). This is when we take the same previous question with any ideal *K,* not necessary cyclic.

Finally in Section 4, We characterize the *WAG2*-absorbing submodules of the R-module $M = M_1 \oplus M_2$, where $R = R_1 \oplus R_2$. We investigate the conditions that make the submodule $N \oplus 0$ *WAG2*-absorbing.

2. The relation between weakly almost generalized 2-absorbing submodules and classes of primary submodules:

We start by the following definitions:

Definition 2.1 (Badawi and Darani, 2013; Badawi, 2007) A proper ideal I of R is weakly 2-absorbing (2-absorbing) ideal if for $a,b,c \in R$ with $0 \neq abc \in I$ ($abc \in I$) either $ab \in I$ or $ac \in I$ or $bc \in I$.

The following definition gives a generalization of weakly 2-absorbing (2-absorbing) ideals to modules.

Definition 2.2 (Darani and Soheilnia, 2011). A proper submodule N of M is called weakly 2-absorbing (2-absorbing) submodule, if for $a,b \in R$ and $m \in M$ with $0 \neq abm \in N$ ($abm \in N$), then $ab \in (N:M)$ or $am \in N$ or $bm \in N$.

Another generalization of weakly 2-absorbing (2-absorbing) ideals was introduced in (Chinwarakorn and Pianskool, 2015). As in following definitions.

Definition 2.3 A proper ideal I of R is *WAG2*-absorbing (*AG2*-absorbing) ideal if for $a,b,c \in R$ with $0 \neq abc \in I$ ($abc \in I$) either $ab \in \sqrt{I}$ or $a^i c \in I$ or $b^j c \in I$ for some positive integers i and j.

Definition 2.4 . A proper submodule N of M is called *WAG2*-absorbing (*AG2*-absorbing) submodule, if for $a,b \in R$ and $m \in M$ with $0 \neq abm \in N$ ($abm \in N$) either $ab \in \sqrt{(N:M)}$ or $a^i m \in N$ or $b^j m \in N$ for some positive integers i and j.

Definition 2.5 (Anderson and Smith, 2003) A proper ideal I of R is called weakly prime (prime) ideal, when from $0 \neq ab \in I$ ($ab \in I$) for some $a,b \in R$, we can conclude that either $a \in I$ or $b \in I$.

Definition 2.6 (McCasland and Moore, 1992; Lu.C.P, 1995; Azizi, 2007)A proper submodule N of M is called weakly prime (prime), when from $0 \neq rx \in N$ ($rx \in N$) for some $r \in R$ and $x \in M$, we can conclude that either $x \in N$ or $rM \subseteq N$.

Definition 2.7 (Atani and Farzalipour, 2005) A proper ideal I of R is called weakly primary (primary) ideal, when from $0 \neq ab \in I$ ($ab \in I$) for some $a,b \in R$, we can conclude that either $a \in I$ or $b \in \sqrt{I}$.

Definition 2.8 (Atani and Frazalipour, 2005; Ashour, 2011) A proper submodule N of M is called weakly primary (primary), when from $0 \neq rx \in N$ ($rx \in N$) for some $r \in R$ and $x \in M$, we can conclude that either $x \in N$ or $r \in \sqrt{(N:M)}$.

In (Behboodi and Koohy, 2004), the following definition was given, it was defined as weakly prime, we will define it as weak prime since there is another definition for weakly prime.

Definition 2.9 A proper submodule N of M is called weak prime submodule, if for $a,b \in R$ and $m \in M$ with $abm \in N$, then $am \in N$ or $bm \in N$.

As a generalization of weak prime submodule, we can define weak primary submodule as in the following definition.

Definition 2.10 A proper submodule N of M is called weak primary submodule, if for $a,b \in R$ and $m \in M$ with $abm \in N$ either $a^i m \in N$ or $b^j m \in N$ for some positive integers i and j.

Remark 2.11 It is easy to see that:

1) Prime submodule → Primary submodule → weakly primary submodule → *WAG2*-absorbing submodule.
2) Primary submodule → weak primary submodule → *AG2*-absorbing submodule → *WAG2*-absorbing submodule.

The following example shows that *WAG2*-absorbing submodules are not weakly primary.

Example 2.12 Let $R = K[X,Y]$ (the ring of polynomials when X and Y are independent and K is a field), $M = R \oplus R$ and $N = \langle Y \rangle \oplus \langle X,Y \rangle$. Then N is *AG2*-absorbing submodule of M. Thus N is *WAG2*-absorbing submodule of M, but it is not weakly primary because $0 \neq X(0,1) \in N$, but neither $X \in \langle Y \rangle = \sqrt{(N:M)}$ nor $(0,1) \in N$.

Also *WAG2*-absorbing submodules are not weak primary as in the following example.

Example 2.13 Let $R = \boldsymbol{Z}$ (the set of integers), $M = \boldsymbol{Z}_{18}$, N = the zero submodule of *M*. Then N is *WAG2*-absorbing submodule of *M*, but it is not weak primary because $0 = 3 \cdot 2 \cdot 3 \in N$, but neither $3^i \cdot 3 \in N$ nor $2^j \cdot 3 \in N$ for any positive integers i, j. Note that $3 \cdot 2 = \sqrt{(N:M)}$.

Proposition 2.14 If a submodule N of an R-module M is an *AG2*-absorbing submodule of M with prime radical ideal $\sqrt{(N:M)}$ of *R*, then N is weak primary submodule of M.

Proof :

Let $a,b \in R$ and with $abm \in N$, then N is *AG2*-absorbing submodule of M implies that either $ab \in \sqrt{(N:M)}$ or $a^i m \in N$ or $b^j m \in N$ for some positive integers i and j. Since $\sqrt{(N:M)}$ is prime, then $ab \in \sqrt{(N:M)}$ implies that $a \in \sqrt{(N:M)}$ or $b \in \sqrt{(N:M)}$. So either $a^s M \subseteq N$ or $b^k M \subseteq N$ for some positive integers s and k. Hence either $a^s m \in N$ or $b^k m \in N$ for some positive integers s and k. These cases give the result that N is weak primary submodule of M.

Since the radical ideal of primary submodule N is prime (Naderi, 2015), then by Remark 2.11 and Proposition 2.14, we can conclude the following result.

Corollary 2.15 Let N be a primary submodule of an R-module M. Then N is weak primary if and only if N is *AG2*-absorbing submodule of *M*.

3. *WAG2*-absorbing submodules and their radical ideals:

Lemma 3.1 Let N be a *WAG2*-absorbing submodule of an R-module *M*. Let $a,b \in R$. If for some cyclic submodule K of *M*, $abK \subseteq N$ and $0 \neq 2abK$, then $ab \in \sqrt{(N:M)}$ or $a^i K \subseteq N$ or $b^j K \subseteq N$ for some positive integers i and j.

Proof Let $ab \notin \sqrt{(N:M)}$. It is enough to prove that $a^i K \subseteq N$ or $b^j K \subseteq N$ for some positive integers i and j. Let $K = \langle z \rangle$. If $0 \neq abz$, then as N is a *WAG2*-absorbing submodule and $ab \notin \sqrt{(N:M)}$, then either $a^i z \in N$ or $b^j z \in N$ for some positive integers i and j.

Now, let $0 = abz$. Since $0 \neq 2abK$, then there exists $x \in K$ such that $0 \neq 2abx$ and so $0 \neq abx \in N$. Since $ab \notin \sqrt{(N:M)}$, either $a^i x \in N$ or $b^j x \in N$ for some positive integers i and j. Put $y = x + z$. Then $0 \neq aby \in N$ and since $ab \notin \sqrt{(N:M)}$, then either $a^t y \in N$ or $b^s y \in N$ for some positive integers t and s.

Thus we consider the three cases:

Case1: $a^r x \in N$ and $b^l x \in N$ for some positive integers r and l. Note that $a^t y \in N$ or $b^s y \in N$ for some positive integers t and s. Let $i = \max\{r,t\}$ and $j = \max\{s,l\}$, then either $a^i z = a^i(y-x) \in N$ or $b^j z = b^j(y-x) \in N$.

Case2: $a^r x \in N$ for some positive integer r and $b^l x \notin N$ for any positive integer l. On the contrary, let $a^k z \notin N$ for any positive integer k. Then $a^t y \notin N$ for any positive integer t, because if $a^{t_0} y \in N$ for some positive integer t_0, then $a^c z \in N$ where $c = \max\{r,t_0\}$, and so $b^s y \in N$ for some positive integer s. Hence $a^t(x+y) \notin N$ for any positive integer t and $b^j(x+y) \notin N$ for any positive integer j. Now, N is a *WAG2*-absorbing submodule and $ab \notin \sqrt{(N:M)}$ implies that
$ab(x+y) = 0$.

But $ab(x+y) = ab(x+x+z) = 2abx + abz = 2abx \neq 0$, which is a contradiction. Thus $a^i z \in N$ for some positive integer i.

Case3: $a^r x \notin N$ for any positive integer r and $b^l x \in N$ for some positive integer l. Then in a similar manner to the proof of case2, we can show that $b^j z \in N$ for some positive integer j.

Lemma 3.2 Let J be an ideal of R, K a cyclic submodule and N a submodule of an R-module M such that $aJK \subseteq N$ where $a \in R$. If N is *WAG2*-absorbing submodule of M, and $0 \neq 4aJK$, then $aJ \subseteq \sqrt{(N:M)}$ or $a \in \sqrt{(N:K)}$ or $J \subseteq \sqrt{(N:K)}$.

Proof Let $aJ \nsubseteq \sqrt{(N:M)}$, Then $aj \notin \sqrt{(N:M)}$ for some $j \in J$.

Claim: there exists $b \in J$ such that $0 \neq 4abK$ and $ab \notin \sqrt{(N:M)}$.

Since $0 \neq 4aJK$, for some $c \in J$, $0 \neq 4acK$. If $ac \notin \sqrt{(N:M)}$ or $0 \neq 4ajK$, then by putting $b = c$ or $b = j$, we get the result. Therefore, let $ac \in \sqrt{(N:M)}$ and $0 = 4ajK$. Hence, $0 \neq 4a(j+c)K \subseteq N$ and $a(j+c) \notin \sqrt{(N:M)}$. Consequently, we find $b = j + c \in J$ such that $0 \neq 4abK$ and $ab \notin \sqrt{(N:M)}$.

Thus $0 \neq 2abK$ and by Lemma 3.1, $a^i K \subseteq N$ or $b^j K \subseteq N$ for some positive integers i and j. Thus $a \in \sqrt{(N:K)}$ or $b \in \sqrt{(N:K)}$. If $a \in \sqrt{(N:K)}$, there is nothing to prove. So assume that $a \notin \sqrt{(N:K)}$ and $b \in \sqrt{(N:K)}$. Now, we show that $aJ \nsubseteq \sqrt{(N:M)}$ or $J \subseteq \sqrt{(N:K)}$.

Let $d \in J$. If $0 \neq 2adK$, then by Lemma 3.1, $ad \in \sqrt{(N:K)}$ or $a^i K \subseteq N$ or $d^j K \subseteq N$ for some positive integers i and j. Since by assumption $a \notin \sqrt{(N:K)}$, then either $ad \in \sqrt{(N:K)}$ or $d \in \sqrt{(N:K)}$. Thus $aJ \subseteq \sqrt{(N:M)}$ or $J \subseteq \sqrt{(N:K)}$.

Next, assume that $0 = 2adK$. Then $0 \neq 2a(b+d)K \subseteq N$ and Lemma 3.1 implies that either $a(b+d) \in \sqrt{(N:K)}$ or $a^i K \subseteq N$ or $(b+d)^j K \subseteq N$ for some positive integers i and j. Since $a \notin \sqrt{(N:K)}$, then either $a(b+d) \in \sqrt{(N:K)}$ or $(b+d) \in \sqrt{(N:K)}$. If $(b+d) \in \sqrt{(N:K)}$, then $d \in \sqrt{(N:K)}$ because $b \in \sqrt{(N:K)}$. Let $a(b+d) \in \sqrt{(N:K)}$ and $(b+d) \notin \sqrt{(N:K)}$. Consider $2a(b+d+b)K = 4abK \neq 0$ and $a(b+d+b)K \subseteq N$. Since $ab \notin \sqrt{(N:M)}$ and $a(b+d) \in \sqrt{(N:K)}$, then $a(b+d+b) \notin \sqrt{(N:M)}$. Thus by Lemma 3.1, $a^i K \subseteq N$ or $(b+d+b)^j K \subseteq N$ for some positive integers i and j. Thus $a \in \sqrt{(N:K)}$ or $b+d+b \in \sqrt{(N:K)}$. However, since $(b+d) \notin \sqrt{(N:K)}$ and $b \in \sqrt{(N:K)}$, then $b+d+b \notin \sqrt{(N:K)}$. Therefore, $a \in \sqrt{(N:K)}$, which is impossible. Hence $(b+d) \in \sqrt{(N:K)}$ and since $b \in \sqrt{(N:K)}$, then $d \in \sqrt{(N:K)}$. Consequently $aJ \subseteq \sqrt{(N:M)}$ or $J \subseteq \sqrt{(N:K)}$ and as $aJ \nsubseteq \sqrt{(N:M)}$, then $J \subseteq \sqrt{(N:K)}$.

Theorem 3.3 Let I, J be ideals of R, K a cyclic submodule and N a submodule of an R-module M. If N is *WAG2*-absorbing submodule of M, $0 \neq IJK \subseteq N$ and $0 \neq 8(IJ + (I+J)(N:M))(K+N)$, then $IJ \subseteq \sqrt{(N:M)}$ or $I \subseteq \sqrt{(N:K)}$ or $J \subseteq \sqrt{(N:K)}$. In particular this holds in the group (M , +) has no elements of order 2.

Proof Note that

$$0 \neq 8(IJ + (I+J)(N:M))(K+N) = 8IJK + 8IJN + 8I(N:M)K + 8I(N:M)N + 8J(N:M)K + 8J(N:M)N.$$

Therefore, one of the following different types is satisfied.

i) $0 \neq 8IJK$. Then for some $a \in J$, we have $0 \neq 8aIK$. Then $0 \neq 4aIK$ and by Lemma 3.2, $aI \subseteq \sqrt{(N:M)}$ or $a \in \sqrt{(N:K)}$ or $I \subseteq \sqrt{(N:K)}$. If $I \subseteq \sqrt{(N:K)}$, then we have the result. Therefore, we suppose that $I \nsubseteq \sqrt{(N:K)}$ so $aI \subseteq \sqrt{(N:M)}$ or $a \in \sqrt{(N:K)}$. Now , we show that $IJ \subseteq \sqrt{(N:M)}$ or $J \subseteq \sqrt{(N:K)}$. To see this, let $c \in J$. If $0 \neq 4cIK$, then by Lemma 3.2, since $I \not\subset \sqrt{(N:K)}$, $cI \subseteq \sqrt{(N:M)}$ or $c \in \sqrt{(N:K)}$. Now, let $4cIK = 0$. So $0 \neq 4(a+c)IK \subseteq N$. Thus by Lemma 3.2, since $I \nsubseteq \sqrt{(N:K)}$, $(a+c)I \subseteq \sqrt{(N:M)}$ or $(a+c) \in \sqrt{(N:K)}$. We consider the following cases.

Case1: $(a+c)I \subseteq \sqrt{(N:M)}$ and $aI \subseteq \sqrt{(N:M)}$. Then $cI \subseteq \sqrt{(N:M)}$.

Case2: $(a+c) \in \sqrt{(N:K)}$ and $a \in \sqrt{(N:K)}$. Hence $c \in \sqrt{(N:K)}$.

Case3: $a \in (\sqrt{(N:M)}:I) \setminus \sqrt{(N:K)}$ and $a+c \in \sqrt{(N:K)} \setminus (\sqrt{(N:M)}:I)$. Therefore, $a+c+a \notin (\sqrt{(N:M)}:I)$ and $a+c+a \notin \sqrt{(N:K)}$. We Consider $4(a+c+a)IK = 8aIK \neq 0$. Hence by Lemma 3.2, since $I \not\subset \sqrt{(N:K)}$, $(a+c+a)I \subseteq \sqrt{(N:M)}$ or $(a+c+a) \in \sqrt{(N:K)}$. So $a+c+a \in (\sqrt{(N:M)}:I)$ or $a+c+a \in \sqrt{(N:K)}$, which is impossible. Therefore, as $aI \subseteq \sqrt{(N:M)}$ or $a \in \sqrt{(N:K)}$ and $(a+c)I \subseteq \sqrt{(N:M)}$ or $(a+c) \in \sqrt{(N:K)}$, one of the following holds:

a) $a \in \sqrt{(N:K)}$ and $a+c \in \sqrt{(N:K)} \setminus (\sqrt{(N:M)}:I)$, which implies $c \in \sqrt{(N:K)}$.

b) $a \in (\sqrt{(N:M)}:I) \setminus \sqrt{(N:K)}$ and $(a+c)I \subseteq \sqrt{(N:M)}$, which implies $cI \subseteq \sqrt{(N:M)}$.

Case4: $a+c \in (\sqrt{(N:M)}:I) \setminus \sqrt{(N:K)}$ and $a \in \sqrt{(N:K)} \setminus (\sqrt{(N:M)}:I)$. Similar to case3, we get $c \in \sqrt{(N:K)}$ or $cI \subseteq \sqrt{(N:M)}$. Consequently, $IJ \subseteq \sqrt{(N:M)}$ or $J \subseteq \sqrt{(N:K)}$.

ii) If $0 \neq 8IJN$ and $8IJK = 0$, then $0 \neq 8IJ(K+N) \subseteq N$, and then by part i), $IJ \subseteq \sqrt{(N:M)}$ or $I \subseteq \sqrt{(N:K)}$ or $J \subseteq \sqrt{(N:K)}$.

iii) Let $0 \neq 8J(N:M)K$ and $8IJK = 0$. Then $8J(I+(N:M))K = 8J(N:M)K \neq 0$ and so according to part i), $J(I+(N:M)) \subseteq \sqrt{(N:M)}$ or $(I+(N:M)) \subseteq \sqrt{(N:K)}$ or $J \subseteq \sqrt{(N:K)}$ so either , $IJ \subseteq \sqrt{(N:M)}$ or $I \subseteq \sqrt{(N:K)}$ or $J \subseteq \sqrt{(N:K)}$. Similarly, if $0 \neq 8I(N:M)K$, we get the result.

iv) Let $0 \neq 8J(N:M)N$ $0 = 8IJK = 8IJN = 8J(N:M)K = 8I(N:M)K.$ Then $8J(I+(N:M))(K+N) = 8J(N:M)N \neq 0$ and so part i) implies that $J(I+(N:M)) \subseteq \sqrt{(N:M)}$ or $(I+(N:M)) \subseteq \sqrt{(N:(K+N))}$ or $J \subseteq \sqrt{(N:(K+N))}$ so either, $IJ \subseteq \sqrt{(N:M)}$ or $I+(N:M) \subseteq \sqrt{(N:K)}$ (which implies that $I \subseteq \sqrt{(N:K)}$) or $J \subseteq \sqrt{(N:K)}$). Similarly, if $0 \neq 8I(N:M)N$, we get the result.

For the particular case, suppose that the group (M , +) has no subgroups of order 2. Then by (Moradi and Azizi, 2016), $0 \neq 8IJK$ and so part i) gives the result.

The following result is the ring version of Lemma 3.1, Lemma 3.2 and Theorem 3.3, for the proof just consider $M = R$.

Corollary 3.4 Let $a, b \in R$, K a cyclic ideal of R and I, J ideals of R and suppose that L is *WAG2*-absorbing ideal of R.

a) If $0 \neq 2abI$ and $abI \subseteq L$, then $ab \in \sqrt{L}$ or $a \in \sqrt{(L:I)}$ or $b \in \sqrt{(L:I)}$.

b) If $0 \neq 4aIJ$ and $aIJ \subseteq L$, then $IJ \in \sqrt{L}$ or $a \in \sqrt{(L:I)}$ or $aJ \in \sqrt{L}$.

c) If $0 \neq IJK \subseteq L$ and $8(IJ(K+L)+IK(J+L)+JK(I+L)+IL(J+K)+JL(I+K)+KL(I+J)+L^2(I+J+K)) \neq 0$, then $IJ \subseteq \sqrt{L}$ or $IK \subseteq \sqrt{L}$ or $JK \subseteq \sqrt{L}$. In particular this holds in the group (R , +) has no elements of order 2.

Now, we recall the following definition, see (Ashour, 2010)

Definition 3.6 Let M be an R-module. The radical annihilator of M is denoted by $rann(M)$ and is defined by

$rann(M) = \{r \mid r \in R \text{ and } r^n M = 0, \text{ for some positive integer } n\}$.

Theorem 3.5 Let M be a cyclic R-module, then the radical ideal of a *WAG2*-absorbing submodule of M is a *WAG2*-absorbing ideal if $rann(M) = 0$.

Proof Let N be a *WAG2*-absorbing submodule an R-module M. First, note that the ideal $rann(M)$ is *WAG2*-absorbing ideal of R. Let $0 \neq abc \in \sqrt{(N:M)}$ with $ab \notin \sqrt{(N:M)}$. Then $(abc)^t M \subseteq N$ for some positive integer t . Since $rann(M) = 0$ and $0 \neq abc$, then $0 \neq (abc)^t M \subseteq N$. Let $M = \langle z \rangle$, then $0 \neq (abc)^t z \in N$.

Thus $0 \neq a^t b^t (c^t z) \in N$. Since N is *WAG2*-absorbing submodule and $ab \notin \sqrt{(N:M)}$ implies that $(ab)^t \notin \sqrt{(N:M)}$, then either $(a^t)^i (c^t z) \in N$ or $(b^t)^j (c^t z) \in N$ for some positive integers i and j. Thus either $(a^i c)^t z \in N$ or $(b^j c)^t z \in N$ for some positive integers i and j. Since $M = \langle z \rangle$, then either $(a^i c)^t M \subseteq N$ or $(b^j c)^t M \subseteq N$ for some positive integers i and j. So, either $a^i c \in \sqrt{(N:M)}$ or $b^j c \in \sqrt{(N:M)}$ for some positive integers i and j. Therefore, $\sqrt{(N:M)}$ is a *WAG2*-absorbing ideal.

4. *WAG2*-absorbing submodules in direct sum of modules:

First, we recall the following two results.

Proposition 4.1 (Chinwarakorn and Pianskool, 2015) Let M be a multiplication R-module. If N is a *WAG2*-absorbing submodule of M , but not *AG2*-absorbing submodule of M, then $N^3 = 0$.

Lemma 4.2 (Moradi and Azizi, 2016)Let K^* be a proper submodule of an R^*-module M^* and $I^* M^* \neq 0$, where I^* is an ideal of R^*. Then there exist $r \in I^*$ and $x \in (M^* \setminus K^*)$ with $rx \neq 0$.

Lemma 4.3 Let $M = M_1 \oplus M_2$ be an R-module of $R = R_1 \oplus R_2$. Let N_1 be submodule of M_1 and N_2 be submodule of M_2.

a) If $N_1^{\,3} \neq 0$ or $M_2^{\,3} \neq 0$ and M is a multiplication R-module, then the following are equivalent:
   i) $N_1 \oplus M_2$ is a *WAG2*-absorbing submodule of the R-module M.
   ii) $N_1 \oplus M_2$ is an *AG2*-absorbing submodule of the R-module M.
   iii) N_1 is an *AG2*-absorbing submodule of the R_1-module M_1.

b) If $N_2^{\,3} \neq 0$ or $M_1^{\,3} \neq 0$ and M is a multiplication R-module, then the following are equivalent:
   i) $M_1 \oplus N_2$ is a *WAG2*-absorbing submodule of the R-module M.
   ii) $M_1 \oplus N_2$ is an *AG2*-absorbing submodule of the R-module M.
   iii) N_2 is an *AG2*-absorbing submodule of the R_2-module M_2.

c) If $N = N_1 \oplus N_2$ is a *WAG2*-absorbing submodule of the R-module M with $N_1 \neq M_1$ and $N_2 \neq M_2$, then N_1 is a weakly primary submodule of the R_1-module M_1 and N_2 is a weakly primary submodule of the R_2-module M_2, moreover, if $N_2 \neq 0$ ($N_1 \neq 0$), then N_1 is a weak primary submodule of the R_1-module M_1 (N_2 is a weak primary submodule of the R_2-module M_2).

d) If N_1 is a primary submodule of the R_1-module M_1 and N_2 is a primary submodule of the R_2-module M_2, then $N = N_1 \oplus N_2$ is an *AG2*-absorbing submodule of the R-module M.

e) If $N = N_1 \oplus N_2$ is a *WAG2*-absorbing submodule of the R-module M with $N_1 \neq M_1$, $N_2 \neq M_2$ and $(N_2 : M_2)M_2 \neq 0$ ($(N_1 : M_1)M_1 \neq 0$), then N_1 is a primary submodule of the R_1-module M_1 (N_2 is a primary submodule of the R_2-module M_2).

Proof

a) i) $\Rightarrow$ ii) If $N_1 \oplus M_2$ is a *WAG2*-absorbing submodule of the R-module M, but not *AG2*-absorbing submodule of M, then by Proposition 4.1, $(N_1 \oplus M_2)^3$ is zero.

Thus $(0,0) = (N_1^{\,3} \oplus M_2^{\,3})$ and so $N_1^{\,3} = 0$ and $M_2^{\,3} = 0$, which is impossible.

ii) $\Rightarrow$ iii) Let $abm_1 \in N_1$ with $a, b \in R$ and $m_1 \in M_1$. Then $ab(m_1, 0) \in N_1 \oplus M_2$. Since $N_1 \oplus M_2$ is an *AG2*-absorbing submodule, then $ab \in \sqrt{(N_1 \oplus M_2 : M)}$ or $a^i (m_1, 0) \in N_1 \oplus M_2$ or $b^j (m_1, 0) \in N_1 \oplus M_2$ for some positive integers i and j. Hence $(ab)^t M \subseteq N_1 \oplus M_2$, which implies that $(ab)^t M_1 \subseteq N_1$, or $a^i m_1 \in N_1$ or $b^j m_1 \in N_1$ for some positive integers i , j and t. Therefore, $ab \in \sqrt{(N_1 : M_1)}$ or $a^i m_1 \in N_1$ or $b^j m_1 \in N_1$ for some positive integers i , j, which gives the result.

iii) $\Rightarrow$ i) Let $0 \neq ab(m_1, m_2) \in N_1 \oplus M_2$ for $a, b \in R$ and $(m_1, m_2) \in M_1 \oplus M_2$, then $abm_1 \in N_1$. Since N_1 is an *AG2*-absorbing submodule, then either $ab \in \sqrt{(N_1 : M_1)}$ or $a^i m_1 \in N_1$ or $b^j m_1 \in N_1$ for some positive integers i, j. If $ab \in \sqrt{(N_1 : M_1)}$, then $(ab)^t M_1 \subseteq N_1$ for some positive integer t, which implies that $(ab)^t (M_1 \oplus M_2) \subseteq N_1 \oplus M_2$ for some positive integer t, so $ab \in \sqrt{(N_1 \oplus M_2 : M)}$. If $a^i m_1 \in N_1$ or $b^j m_1 \in N_1$ for some positive integers i, j, then $a^i (m_1, m_2) \in N_1 \oplus M_2$ or $b^j (m_1, m_2) \in N_1 \oplus M_2$ for some positive integers i, j. Hence $N_1 \oplus M_2$ is a *WAG2*-absorbing submodule.

b) The proof of part b is similar to the proof of part a.

c) Let $0 \neq rm_1 \in N_1$ where $r \in R$ and $m_1 \in M_1$. Let $m_2 \in M_2 / N_2$. Then $(0,0) \neq (1,0)(r,1)(m_1, m_2) \in N_1 \oplus N_2$. Since $N = N_1 \oplus N_2$ is a *WAG2*-absorbing submodule of the *R*-module *M,* then either $(1,0)(r,1) = (r,0) \in \sqrt{(N : M)}$ or $(1,0)^t (m_1, m_2) = (m_1, 0) \in N$ or $(r,1)^s (m_1, m_2) = (r^s m_1, m_2) \in N$ for some positive integers t, s. If $(r,0) \in \sqrt{(N : M)}$, then $(r,0)^k M \subseteq N$ for some positive integer k, which implies that $r^k M_1 \subseteq N_1$ for some positive integer k, so $r \in \sqrt{(N_1 : M_1)}$. If $(m_1, 0) \in N$ for some positive integer t, then $m_1 \in N_1$. If $(r^s m_1, m_2) \in N$ for some positive integer s, then $r^s m_1 \in N_1$ for some positive integer s and $m_2 \in N_2$, which is a contradiction to the assumption that $m_2 \in M_2 / N_2$. Thus either $r \in \sqrt{(N_1 : M_1)}$ or $m_1 \in N_1$. This shows that N_1 is a primary submodule of the R_1-module M_1.
In a similar way we can show that N_2 is a weakly primary submodule of the R_2-module M_2.
Now, let $N_2 \neq 0$ and suppose that $aby_1 \in N_1$ with $a, b \in R$ and $y_1 \in M_1$. Let $0 \neq y_2 \in N_2$. Then $(0,0) \neq (a,1)(b,1)(y_1, y_2) \in N_1 \oplus N_2$. Since $N = N_1 \oplus N_2$ is a *WAG2*-absorbing submodule of the *R*-module *M,* then either $(a,1)(b,1) = (ab,1) \in \sqrt{(N : M)}$ or $(a,1)^t (y_1, y_2) = (a^t y_1, y_2) \in N$ or $(b,1)^s (y_1, y_2) = (b^s y_1, y_2) \in N$ for some positive integers t, s. If $(ab,1) \in \sqrt{(N : M)}$, then $(ab,1)^k M \subseteq N$ for some positive integer k, which implies that $((ab)^k, 1)(M_1 \oplus M_2) \subseteq N_1 \oplus N_2$ for some positive integer k, then $M_2 \subseteq N_2$, which is a contradiction. If $(a^t y_1, y_2) \in N$ for some positive integer t, then $a^t y_1 \in N_1$. If $(b^s y_1, y_2) \in N$ for some positive integer s, then $b^s y_1 \in N_1$ for some positive integer s. Thus either $a^t y_1 \in N_1$ or $b^s y_1 \in N_1$ for some positive integers t, s. This shows that N_1 is a weak primary submodule of the R_1-module M_1.
In a similar way if we consider the condition that $N_1 \neq 0$, then we can show that N_2 is a weakly primary submodule of the R_2-module M_2.

d) Let $(a,c), (b,d) \in R$ and $(m_1, m_2) \in M$ with $(a,c)(b,d)(m_1, m_2) \in N$. Then $abm_1 \in N_1$ and $cdm_2 \in N_2$. Since N_1 is a primary submodule of the R_1-module M_1, then $ab \in \sqrt{(N_1 : M_1)}$ or $m_1 \in N_1$. Similarly, N_2 is a primary submodule of the R_2-module M_2 implies that either $cd \in \sqrt{(N_2 : M_2)}$ or $m_2 \in N_2$. Since N_1 and N_2 are primary, then $\sqrt{(N_1 : M_1)}$ and $\sqrt{(N_2 : M_2)}$ are prime ideals of R_1 and R_2, respectively, see (Naderi, 2014). So N_1 is primary implies that $a \in \sqrt{(N_1 : M_1)}$ or $b \in \sqrt{(N_1 : M_1)}$ or $m_1 \in N_1$. Similarly, N_2 is primary implies that $c \in \sqrt{(N_2 : M_2)}$ or $d \in \sqrt{(N_2 : M_2)}$ or $m_2 \in N_2$. In any of these cases, we get $(a,c)(b,d) \in \sqrt{(N : M)}$ or $(a,c)^i (m_1, m_2) \in N$ or $(b,d)^j (m_1, m_2) \in N$ for some positive integers i, j. Hence $N = N_1 \oplus N_2$ is an *AG2*-absorbing submodule of the *R*-module M.

e) Let $rx \in N_1$ where $r \in R$ and $x \in M_1$. Apply Lemma 4.2 for $R^* = R_2$, $I^* = (N_2 : M_2)$, $K^* = N_2$

and $M^* = M_2$ to see that there exist $s \in (N_2 : M_2)$ and $z \in (M_2 \setminus N_2)$ with $sz \neq 0$. Note that $(0,0) \neq (1,s)(r,1)(x,z) \in N_1 \oplus N_2$ and since $N = N_1 \oplus N_2$ is a *WAG2*-absorbing submodule of the *R*-module *M*, then either $(1,s)(r,1) = (r,s) \in \sqrt{(N:M)}$ or $(1,s)^t(x,z) = (x, s^t z) \in N$ or $(r,1)^s(x,z) = (r^s x, z) \in N$ for some positive integers t, s. As $z \in (M_2 \setminus N_2)$, $(r,1)^s(x,z) = (r^s x, z) \notin N$. Hence $(r,s)^l M \in N$ or $(x, s^t z) \in N$ for some positive integers l, t. Therefore, $r^l M_1 \subseteq N_1$ or $x \in N_1$ for some positive integer l. Hence $r \in \sqrt{(N_1 : M_1)}$ or $x \in N_1$. This implies that N_1 is a primary submodule of the R_1-module M_1.

In a similar way, if we use the condition that $(N_1 : M_1)M_1 \neq 0$, then we can show that N_2 is a primary submodule of the R_2-module M_2.

Theorem 4.4 Let $M = M_1 \oplus M_2$ be an *R*-module of $R = R_1 \oplus R_2$. Let N_1 be submodule of M_1. Let $N_1 \neq M_1$ and $0 \neq M_2$. The submodule $N_1 \oplus 0$ is a *WAG2*-absorbing submodule of the *R*-module *M* if and only if one of the following holds:

i) N_1 is a weakly primary submodule of the R_1-module M_1 , 0 is a primary submodule of the R_2-module M_2 and $(N_1 : M_1)M_1 \neq 0$.

ii) N_1 is a weakly primary submodule of the R_1-module M_1 , 0 is a weak primary submodule of the R_2-module M_2 and $(N_1 : M_1)M_1 = 0$.

iii) $N_1 = 0$.

Proof $\Rightarrow$) Let $N_1 \oplus 0$ be a *WAG2*-absorbing submodule of the *R*-module *M* and $N_1 \neq 0$. Then by Lemma 4.3 part c), N_1 is a weakly primary submodule of the R_1-module M_1. If $(N_1 : M_1)M_1 \neq 0$, then by Lemma 4.3 part e), 0 is a primary submodule of the R_2-module M_2 . If $(N_1 : M_1)M_1 = 0$, since $N_1 \neq 0$, then by Lemma 4.3 part c), 0 is a weak primary submodule of the R_2-module M_2.

$\Leftarrow$) Assume that $N_1 \neq 0$ and $(0,0) \neq (a,b)(c,d)(x,y) \in N_1 \oplus 0$ where $(a,c),(b,d) \in R$ and $(x,y) \in M$. Then $0 \neq acx \in N_1$ and $bdy = 0$. Since N_1 is a weakly primary submodule of the R_1-module M_1, then $a \in \sqrt{(N_1 : M_1)}$ or $cx \in N_1$. If $cx \in N_1$, then $0 \neq cx \in N_1$, otherwise $0 = acx$, and N_1 is a weakly primary submodule of the R_1-module M_1 implies that $c \in \sqrt{(N_1 : M_1)}$ or $x \in N_1$. Therefore, $a \in \sqrt{(N_1 : M_1)}$ or $c \in \sqrt{(N_1 : M_1)}$ or $x \in N_1$. If 0 is a primary submodule of the R_2-module M_2 and $(N_1 : M_1)M_1 \neq 0$, then $bdy = 0$ implies that $bd \in \sqrt{(0 : M_2)}$ or $y = 0$. But, $\sqrt{(0 : M_2)}$ is a prime ideal of R_2, see (El-Atrash and Ashour, 2005). Thus $b \in \sqrt{(0 : M_2)}$ or $d \in \sqrt{(0 : M_2)}$ or $y = 0$. Now, it is easy to check that in any of the above cases, either $(a,b)(c,d) \in \sqrt{(N_1 \oplus 0 : M)}$ or $(a,b)^t(x,y) \in N_1 \oplus 0$ or $(c,d)^s(x,y) \in N_1 \oplus 0$ for some positive integers t, s. Consequently, $N_1 \oplus 0$ is a *WAG2*-absorbing submodule of the *R*-module M.

Now, assume that 0 is a weak primary submodule of the R_2-module M_2 and $\sqrt{(N_1 : M_1)}M_1 = 0$. If $a \in \sqrt{(N_1 : M_1)}$ or $c \in \sqrt{(N_1 : M_1)}$, then $acx \in \sqrt{(N_1 : M_1)}M_1 = 0$, which is impossible. Thus $x \in N_1$. Since $bdy = 0$ and 0 is a weak primary submodule of the R_2-module M_2, then $b^t y = 0$ or $d^s y = 0$ for some positive integers t, s. Therefore, either $(a,b)^t(x,y) \in N_1 \oplus 0$ or $(c,d)^s(x,y) \in N_1 \oplus 0$ for some positive integers t, s. Consequently, $N_1 \oplus 0$ is a weak primary submodule of the *R*-module *M* and by Remark 2.11 $N_1 \oplus 0$ is a *WAG2*-absorbing submodule of the *R*-module M.

Theorem 4.6 Let $M = M_1 \oplus M_2$ be an *R*-module of $R = R_1 \oplus R_2$. Let N_1 be submodule of M_1. Let $N_1 \neq M_1$ and $0 \neq M_2$. If N_1 is a weakly primary submodule of the R_1-module M_1, 0 is a primary submodule of the R_2-module M_2 and $(N_1 : M_1)M_1 \neq 0$, then $N_1 \oplus 0$ is an *AG2*-absorbing submodule of the R-module M if and only if N_1 is a primary submodule of the R_1-module M_1.

Proof ($\Rightarrow$) If N_1 is not primary submodule of the R_1-module M_1, then for some $t \in R_1 \setminus \sqrt{(N_1 : M_1)}$ and $z \in (M_1 \setminus N_1)$ we have $tz \in N_1$. Since N_1 is a weakly primary submodule of the R_1-module M_1, then $tz = 0$. Now choose $0 \neq u \in M_2$. Then $(0,0) = (1,0)(t,1)(z,u) \in N_1 \oplus 0$ with $(1,0)(t,1) \notin \sqrt{(N_1 \oplus 0 : M)}$, $(1,0)^t (z,u) \notin N_1 \oplus 0$ and $(t,1)^s (z,u) \notin N_1 \oplus 0$ for any positive integers t, s. Therefore, $N_1 \oplus 0$ is not an *AG2*-absorbing submodule of the R-module M.

($\Leftarrow$) If N_1 is a primary submodule of the R_1-module M_1, then as 0 is a primary submodule of the R_2-module M_2 by Lemma 4.3 part d), $N_1 \oplus 0$ is an *AG2*-absorbing submodule of the R-module *M.*

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حول المقاسات الجزئية الضعيفة الممتصة من النوع 2 المعممة تقريبا

ليكن M مقاسا معرفا على الحلقة الابدالية R ذات المحايد غير الصفري. نهتم في هذا البحث بدراسة العلاقة بين فئات مختلفة من المقاسات الجزئية الابتدائية والمقاسات الجزئية الضعيفة الممتصة من النوع 2 المعممة تقريبا مثل المقاسات الابتدائية والمقاسات الابتدائية الضعيفة والمقاسات الابتدائية المضعفة والمقاسات الجزئية الممتصة من النوع 2 المعممة تقريبا. كما نقوم بدراسة الجذور لهذه المقاسات ونبرهن العديد من النتائج الخاصة بها. كذلك نطرح في هذه الدراسة سؤال يعتبر تعميم لسؤال طرحه Badawi, وDarani في 2013 وأجاب عليه Moradi وAzizi في 2016, وهو كالآتي: ليكن L مثالي ضعيف ممتصة من النوع 2 المعمم تقريبا على R بحيث يحقق العلاقة $L \supseteq IJK \neq 0$, حيث I, J وK مثاليات على R. هل من الممكن أن يؤدي ذلك إلى أن $IJ \subseteq \sqrt{L}$ أو $JK \subseteq \sqrt{L}$ أو $IK \subseteq \sqrt{L}$؟ ونجيب في هذا البحث عن السؤال المطروح في حالة كون المثالي K مثالي حلقي. كما ندرس المقاسات الجزئية الضعيفة الممتصة من النوع 2 المعممة تقريبا على المقاس $M = M_1 \oplus M_2$ المعرف على الحلقة $R = R_1 \oplus R_2$.

ونستنتج الشروط الكافية لجعل المقاس الجزئي $N_1 \oplus 0$ مقاسا جزئيا ضعيفا ممتصا من النوع 2 المعمم تقريبا.

كلمات مفتاحية:
المقاسات الجزئية الابتدائية، المقاسات الجزئية الابتدائية الضعيفة،
المقاسات الجزئية الابتدائية المضعفة،
المقاسات الجزئية الممتصة من النوع 2 المعممة تقريبا،
المقاسات الجزئية الضعيفة الممتصة من النوع 2 المعممة تقريبا،
جذور المقاسات الجزئية الضعيفة الممتصة من النوع 2 المعممة تقريبا.